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THE EXISTENCE OF EFFICIENT AND INCENTIVE COMPATIBLE EQUILIBRIA WITH PUBLIC GOODS

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by

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1. Introduction

In the paper "Optimal Allocation of Public Goods..." [1977] we presented an informationally decentralized mechanism for determining public goods allocations which rely on consumers correctly revealing their demands for public goods. The important feature of this mechanism is the fact that if consumers behave competitively in markets for private goods and follow Nash behavior in their choices of messages ("demands") to the mechanism, then, for a wide class of economies, equilibria will be Pareto optimal.

It is now known that other mechanisms for public goods allocation also have the property that their (Nash) equilibria are Pareto optimal. Two such mechanisms are examined in the papers of Hurwicz [1976] and Walker [1977]. Another is the particularly simple one[1/] that chooses a level of public goods equal to the quantity demanded by consumers and assesses each one a constant, arbitrarily fixed, proportional share of the total cost. Although equilibria for this mechanism are efficient, they rarely exist! In this paper we show that, for a wide class of economies, an equilibrium under our mechanism exists and, thus, the Pareto optimality of equilibria is not a vacuous property.

The strongest conjecture one might seek to prove is that equilibria under our mechanism exist whenever the economy has a Lindahl equilibrium.[2/] However, this conjecture is false for an interesting economic reason. The tax rules of our mechanism, which assign cost shares for the public goods provided, may confiscate enough wealth from a consumer to leave him worse off than he would be consuming only his initial endowment.[3/] In extreme cases, his tax may be greater than his wealth and thus may bankrupt him. But this can occur only when too many resources are devoted to the production of public goods. Thus an additional assumption ruling out such cases, along with assumptions sufficient to guarantee Lindahl equilibria exist, suffice to establish existence under our mechanism. The addition assumption is, approximately, that at all Pareto-optimal allocations the amount of private goods consumption is greater than some small but strictly positive amount. Thus, most economies with a Lindahl equilibrium will have an equilibrium under our mechanism as well.

In Section 2, we present the general model of a competitive private ownership economy with a government (or mechanism) and the specific government we developed in [1977]. In Section 4, the existence theorem delineating the economies having equilibria under the rules defining our government is stated and proved. Because the model is general and technically complicated, we present in Section 3 a related existence theorem for simple two good

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(one private-one public) neoclassical economies with bounded consumptions. A heuristic example explaining how the bankruptcy problem can arise under standard assumptions is also given.

2. Competitive Private Ownership Economies with Government

2.1 The Economy

We consider an Arrow-Debreu private ownership economy with public goods and a government. A bundle of L private goods is denoted by x, an element of $\mathbb{R}^L$ (L-dimensional Euclidean space), and a bundle of K public goods is denoted by y, an element of $\mathbb{R}^K$. Prices for private and public goods are denoted by $p \in \mathbb{R}^L$ and $q \in \mathbb{R}^K$ respectively, and a price system for all goods by $s = (p,q) \in \mathbb{R}^{L+K}$.

There are $I \geq 3$ consumers; each characterized by (i) a consumption set $X^i \subseteq \mathbb{R}^{L+K}$, (ii) a preference ordering $\precsim_i$ on X^i, and (iii) an initial endowment of private goods, $\omega^i \in \mathbb{R}^L$. There are J producers; each characterized by a production set $Z^j \subseteq \mathbb{R}^{L+K}$ containing all technologically feasible input-output vectors $z^j = (z_x^j, z_y^j)$. Associated with each producer j is a profit share distribution $<\theta^{ij}>_i$ with $0 \leq \theta^{ij} \leq 1$ and $\Sigma_i \theta^{ij} = 1$ where θ^{ij} is consumer i's share of firm j's profits.

The distinction between private and public goods results from specifying that the total net production of public goods, $\Sigma_j z_y^j = z_y$ is consumed by each consumer whereas that of private goods, $\Sigma_j z_x^j = z_x$, is to be divided among the consumers. Thus:

Definition 2.1: An attainable allocation a is an $(I + 1 + J)$-tuple $\{<x^i>, y, <z^j>\}$[4] where $x^i \in \mathbb{R}^L$, $y \in \mathbb{R}^K$, and $z^j \in \mathbb{R}^{L+K}$ such that

$$
\begin{aligned}
&\text{(i)} && (x^i, y) \in X^i \quad \text{all } i, \\
(2.1)\quad &\text{(ii)} && z^j \in Z^j \quad \text{all } j, \text{ and} \\
&\text{(iii)} && [\Sigma_i (x^i - \omega^i), y] = \Sigma_j z^j.
\end{aligned}
$$

A private ownership economy is denoted by $\mathcal{E} \equiv \{<X^i, \precsim_i, \omega^i>, <Z^j>, <\theta^{ij}>\}$.

2.2 The Government

In a private ownership economy, private goods are purchased by consumers in markets but public goods are purchased in markets only by a special agent--the government. The government must therefore (i) choose the quantity of each public good to purchase and (ii) raise through taxes the necessary funds to finance its purchases. Now, to perform these tasks efficiently, the government needs to obtain information about consumers' preferences. Thus we suppose the consumers communicate messages to the government that the government then uses to determine the public goods quantities and taxes in accordance with some fixed rules.

Formally, a government G, is specified by (i) a language or message space M, an abstract set, containing as elements all possible messages, m^i, each consumer may send, (ii) an allocation rule, $y: M^I \times \mathbb{R}^{L+K} \to \mathbb{R}^K$, which is a function of joint messages $m = (m^1, \ldots, m^I)$ and prices $s = (p,q)$ specifying the quantities of the public goods to be purchased, and (iii) consumer tax rules, $C^i: M^I \times \mathbb{R}^{L+K} \to \mathbb{R}$, that specify each consumer's lump-sum tax as a function, also, of joint messages m and prices s. We may thus denote an arbitrary government by $G = \{M, y(\cdot), <C^i(\cdot)>\}$.

2.3 Producer and Consumer Behavior

As price-taking profit maximizers, each producer j is assumed to choose an input-output vector from his production set Z^j that maximizes $s \cdot z^j$ for given prices s.

Definition 2.2:

(i) The supply correspondence of the j-th firm, ϕ^j: $\mathbb{R}^{L+K} \to \mathbb{R}^{L+K}$ is defined by:

$$\phi^j(s) \equiv \{z^j \in Z^j \mid s \cdot z^j \text{ is maximal over } Z^j\} \;.$$

(ii) The profit function of the j-th firm, π^j: $\mathbb{R}^{L+K} \to \mathbb{R}$ is defined by:

$$\pi^j(s) \equiv s \cdot \phi^j(s) \;.$$

Each consumer must choose a private goods consumption bundle $x^i \in \mathbb{R}^L$ and a message $m^i \in M$ to send the government. We assume consumers take as given the prices of all goods, their wealth, and the messages of all other consumers. They do consider, however, how their message affects the allocation of public goods and their tax. Thus, each chooses a decision pair (x^i,m^i) to maximize preferences over consumption bundles (x^i,y) subject to a budget constraint.

Definition 2.3:

(i) The budget correspondence of the i-th consumer, β^i: $M^{I-1} \times \mathbb{R}^{L+K} \times \mathbb{R} \to \mathbb{R}^L \times M$ is defined by:[5]

$$\beta^i(m^{)i(},s,w^i) \equiv \{(\bar{x}^i,\bar{m}^i) \in \mathbb{R}^L \times M \mid (\bar{x}^i, y(m/\bar{m}^i)) \in X^i,$$
$$p \cdot \bar{x}^i + C^i(m/\bar{m}^i,s) \le w^i\}$$

where w^i is his wealth.

(ii) The decision correspondence of the i-th consumer, δ^i: $M^{I-1} \times \mathbb{R}^{L+K} \times \mathbb{R} \to \mathbb{R}^L \times M$ is defined by:

$$\delta^i(m^{)i(},s,w^i) \equiv \{(\bar{x}^i,\bar{m}^i) \in \beta^i(m^{)i(},s,w^i) \mid (\bar{x}^i,y(m/\bar{m}^i)) \succsim_i (x^i,y(m/\bar{m}^i))$$
$$\text{for all } (x^i,m^i) \in \beta^i(m^{)i(},s,w^i) \;.$$

2.4 Equilibrium

The concept of an equilibrium for this model is a natural generalization of a competitive equilibrium for the private goods only model.

Definition 2.4: A competitive equilibrium under the government G in the private ownership economy E is an $(I + J + 1)$-tuple $\varepsilon = \{\langle x^i,m^i\rangle, \langle z^j\rangle, s\}$ of consumer decisions, producer decisions, and a price system such that:

(i) $(x^i,m^i) \in \delta^i(m^{)i(},s,w^i(s))$ all i (preference maximization) where the wealth of i is: $w^i(s) \equiv p \cdot \omega^i + \Sigma_j \theta^{ij}\pi^j(s)$.

(ii) $z^j \in \phi^j(s)$ all j (profit maximization), and

(iii) $(\Sigma_i(x^i - \omega^i), y(m)) = \Sigma_j z^j$ (supply equals demand).

2.5 The Quadratic Government and Potential Bankruptcy

In our previous paper [1977] this model was developed to examine the so-called "Free Rider Problem." We defined a specific government such that if faced with its particular allocation and tax rules, each consumer would find it in his self-interest to correctly reveal his true demand for the public goods, even though he could falsely report his demand without fear of detection. Both Fundamental Theorems of Welfare Economies were proved: A competitive equilibrium under this government is Pareto-optimal (Non-wastefullness Theorem) and every Pareto-optimal allocation is a competitive allocation following, if necessary, some redistribution of initial endowments (Unbiasedness Theorem).

The particular (class of) government we analyzed, called the Quadratic (Q) government,[6/] is specified by:

$$G^Q = \{M, y(\cdot), \langle C^i(\cdot)\rangle\} \quad \text{where}$$

$$(2.2)\quad \begin{aligned} &\text{(i)} \quad M = \mathbb{R}^K \\ &\text{(ii)} \quad y(m) = \Sigma_h m^h \\ &\text{(iii)} \quad C^i(m,s) = \alpha^i q \cdot \Sigma_h m^h + \frac{\gamma}{2}\left[\frac{I-1}{I}(m^i - \mu^i)^2 - \sigma^{i^2}\right] \end{aligned}$$

where $\Sigma_i \alpha^i = 1$, $\gamma > 0$ are parameters and

$$(2.3)\quad \begin{cases} \mu^i \equiv \mu(m^{)i(}) \equiv \dfrac{1}{I-1}\Sigma_{h\neq i} m^h \\ \sigma^{i^2} = \sigma(m^{)i(})^2 \equiv \dfrac{1}{I-2}\Sigma_{h\neq i}(m^h - \mu^i)^2 \end{cases} .$$

Each consumer's message m^i may be interpreted as his demand (which may be negative) since the allocation is just the sum of all consumers' messages (demands). Each consumer's tax consists of a proportional share of the total cost plus an amount increasing in the squared deviation of his demand from the average of the others' demands and decreasing in the sum of squared deviations of the others' demands from their average. Another interpretation of a consumer's message as reported willingness to pay is provided in Groves and Ledyard [1977].

Remark: It should be noted that both Fundamental Theorems of Welfare Economics remain valid for some variants in the choice of the parameters α^i and γ in the cost rules $C^i(\cdot)$ of G^Q. First of all, α^i and γ can be permitted to depend on the prices s. As specified in (2.2), the cost functions $C^i(\cdot)$ are not homogeneous of degree one in prices and thus, the consumers' decision rules ("demand functions") are not homogeneous of degree zero in prices and income. However, if $\gamma^*(s) = \gamma \cdot \|s\|$ is substituted for γ in (2.2), homogeneity will be assured without affecting the validity of the optimality theorems in [1977] or the existence results presented below. Similarly, parameter α^i may be made dependent on prices s and also on the other agents' messages, $m^{)i(}$, and other potentially observable data of the model such as endowments. The only constraint is the equality $\Sigma_i \alpha^i = 1$ which must be satisfied, at least in equilibrium. In proving existence below we consider the variant in which α^i is the proportion of agent i's wealth to aggregate wealth.

3. Existence and Optimality for Two Good Neo-Classical Economies with Bounded Consumption[7/]

3.1 The Simple Economy

In this model, we assume there are only two goods, one public, y, and one private, x, $(L = K = 1)$. We consider a single producer $(J = 1)$ with the linear technology, $z_y = -cz_x$ for $z_x \leq 0$. Without loss in generality, let $c = 1$. Thus $Z = \{z \in \mathbb{R}^2 | z_y = -z_x,\ z_x \leq 0\}$ and the only relative price q/p consistent with profit maximization for this technology is unity (1) at which no profits are realized. We therefore set the price ratio q/p to unity and do not need to specify a profit share distribution. The $I \geq 3$ consumers in this model are characterized by (i) their consumption sets $X^i = \mathbb{R}^2_+$, (ii) their initial endowment of the private good $w^i > 0$, and (iii) their preferences $\precsim_i$ which are assumed to be representable on X^i by a strictly increasing, strictly quasi-concave, smooth (i.e. continuous second derivatives) utility function $u^i(\cdot,\cdot)$ defined over the entire space $\mathbb{R}^2$. (In other words, we assume there exists a function $u^i(\cdot,\cdot)$ defined on $\mathbb{R}^2$ with these properties such that on $X^i = \mathbb{R}^2_+$ the function represents i's preferences.)

Let $E = \{<\mathbb{R}^2_+, u^i(\cdot), w^i>\}$ denote this economy. An attainable allocation a for E is an $(I + 1)$-tuple $(<x^i>, y) \in \mathbb{R}^{n+1}$ such that

$$\text{(3.1)}\quad \begin{aligned} &\text{(i)}\quad x^i \geq 0,\ y \geq 0,\ \text{i.e.}\ (x^i, y) \in X^i \ \text{all } i. \\ &\text{(ii)}\quad y = \Sigma_i(w^i - x^i),\ \text{i.e.}\ z_y = -z_x \ \text{where}\ z_y = y \ \text{and} \\ &\qquad z_x = \Sigma_i(x^i - w^i). \end{aligned}$$

Furthermore, it is straightforward to show:

Proposition 1: An allocation $a = (<x^i>, y) \in \mathbb{R}^{n+1}_+$ is Pareto-optimal (i.e. efficient) if but not only if

$$\text{(3.2)}\quad \begin{aligned} &\text{(i)}\quad \Sigma_i x^i + y = \Sigma_i w^i, \text{ and} \\ &\text{(ii)}\quad \Sigma_i \pi^i(x^i, y) = 1 \quad \text{where} \quad \pi^i(\cdot,\cdot) \equiv (\partial u^i/\partial y)/(\partial u^i/\partial x^i) \end{aligned}$$

is i's marginal rate of substitution of the private for the public good. Note that while (i) is also necessary for efficiency since it is required for any attainable allocation, (ii) is not since there may exist boundary (i.e. where $x^i = 0$) Pareto optima not satisfying (ii).

3.2 The Quadratic Government

Under the Quadratic government G^Q, defined by (2.2) above, a competitive equilibrium for the economy E is given simply by a joint message $\hat{m}$ such that:

$$\text{(3.3)}\quad \begin{aligned} &\hat{m}^i \quad \text{maximizes} \quad u^i[x^i(\hat{m}/m^i), \Sigma_{h\neq i}\hat{m}^h + m^i] \\ &\qquad \text{subject to} \quad (x^i(\hat{m}/m^i), \Sigma_{h\neq i}\hat{m}^h + m^i) \geq 0 \\ &\qquad \text{and} \quad x^i(\hat{m}/m^i) = w^i - C^i(\hat{m}/m^i) \ . \end{aligned}$$

(It is easy to see that the $(I + 2)$-tuple $(<\hat{x}^i, \hat{m}^i>, \hat{z}, \hat{s})$ where $\hat{x}^i \equiv x^i(\hat{m})$, $\hat{z} \equiv (\Sigma_i(\hat{x}^i - w^i), y(\hat{m}))$, and $\hat{s} = \underline{1}$ satisfies Definition 2.4 under our general assumptions on E.)[8/]

It may be observed that if the non-negativity constraints are not binding in (3.3), then the competitive equilibrium joint message is just a Nash equilibrium of the n-person game defined by the government

$G^Q = \{M, y(\cdot), <C^i(\cdot)>\}$ and economy $E = \{<\mathbb{R}^2_+, u^i(\cdot), w^i>\}$ as follows, in normal form:

$$F^Q = \{I, <M^i>, <v^i(\cdot)>\} \tag{3.4}$$

where $I \equiv \{1,\ldots,I\}$ is the player set, $M^i \equiv M$ is player i's strategy space, and $v^i(\cdot) \equiv u_i(x^i(\cdot), y(\cdot))$ is i's payoff function where $x^i(\cdot) = w^i - C^i(\cdot)$. Note that by assuming $u^i(\cdot)$ is defined over all of $\mathbb{R}^2$, the payoff function $v^i(\cdot)$ are well-defined over the space M^I of all joint strategies m.

Thus, every interior competitive equilibrium (i.e. one at which the non-negativity constraints are not tight) is defined by a Nash equilibrium $\hat{m}$ at which $y(\hat{m})$ and $x^i(\hat{m})$, for all i, are non-negative (i.e. the allocation is individually feasible). Furthermore, if the economy E is such that all Nash equilibria of F^Q yield individually feasible allocations, then every competitive equilibrium is interior and is defined by a Nash equilibrium. However, for general economies E, not every Nash equilibrium will yield a competitive equilibrium and not every competitive equilibrium is defined by a Nash equilibrium.

3.3 Optimality of an Interior Equilibrium

In order to prove that every interior competitive equilibrium under G^Q is efficient we use the following characterization of the Nash equilibria of F^Q.

Proposition 2: A joint message $\hat{m}$ is a Nash Equilibrium (N.E.) of the game F^Q if and only if

$$\pi^i[x^i(\hat{m}), y(\hat{m})] = \alpha^i + \frac{\gamma(I-1)}{I}(\hat{m}^i - \hat{\mu}^i) \quad \text{all } i \ . \tag{3.5}$$

Proof: $\hat{m}$ is a N.E. if and only if for every i, $\hat{m}^i$ maximizes $v^i(\hat{m}/m^i) = u^i[x^i(\hat{m}/m^i), y(\hat{m}/m^i)]$. Under our general assumptions on u^i, the First Order Conditions are both necessary and sufficient for $\hat{m}^i$ to maximize $v^i(\hat{m}/m^i)$. Thus

$$\left.\frac{\partial v^i}{\partial m^i}\right|_{\hat{m}} = \frac{\partial u^i}{\partial x^i}\cdot\frac{\partial x^i}{\partial m^i} + \frac{\partial u^i}{\partial y}\cdot\frac{\partial y}{\partial m^i} = \frac{\partial u^i}{\partial x^i}\left(-\alpha^i - \frac{\gamma(I-1)}{I}(\hat{m}^i - \hat{\mu}^i)\right) + \frac{\partial u^i}{\partial y} = 0 \tag{3.6}$$

or

$$\left.\frac{\partial u^i/\partial y}{\partial u^i/\partial x^i}\right|_{\hat{m}} = \pi^i[x^i(\hat{m}), y(\hat{m})] = \alpha^i + \frac{\gamma(I-1)}{I}(\hat{m}^i - \hat{\mu}^i) \ .$$

The First Welfare Theorem (Non-wastefulness) may now be proved for those economies E, under the Quadratic government, for which all competitive equilibria are interior.

Theorem 1: Every interior competitive equilibrium under G^Q for the economy E is Pareto-optimal.

Proof: Since every interior competitive equilibrium is defined by a N.E. of F^Q at which the allocation is individually feasible, it suffices to show that the allocation defined by any N.E. $\hat{m}$ is efficient if it is also individually feasible. By Proposition 1 we thus only need show $\Sigma_i \pi^i[x^i(\hat{m}), y(\hat{m})] = 1$, since $\Sigma_i x^i(\hat{m}) + y(\hat{m}) = \Sigma_i w^i$ by the definition of

$x^i(m)$ as $\Sigma_i C^i(m) = y(m)$ for every joint message m under G^Q. But, by Proposition 2, since $\hat{m}$ satisfies (3.5),

$$\Sigma_i \pi^i[x^i(\hat{m}), y(\hat{m})] = \Sigma\alpha^i + \frac{\gamma(I-1)}{I}\Sigma_i(\hat{m}^i - \hat{\mu}^i) = 1 \quad .$$

3.4 Existence of an Equilibrium

In order to prove the existence of a competitive equilibrium we first prove that the game F^Q has a Nash equilibrium. In addition to the general assumptions on the utility functions already introduced, we make two additional primarily technical assumptions:

Assumption A.1: For every i, $\lim_{y_t \to \infty} \pi^i(x_t^i, y_t) = 0$ if $x_t^i \leq \bar{x}^i < \infty$ all t, where $\bar{x}^i$ is arbitrary.

Assumption A.2: $\Sigma_i \pi^i(x^i, 0) > 1$ for all $x = (x^1, \ldots, x^I) \in \mathbb{R}^I$ such that $\Sigma_i x^i \geq \Sigma_i w^i$.

Assumption A.1 merely requires that if i's consumption of private good is bounded above, then i becomes satiated in the limit with the public good. Since the set of attainable allocations for the economy E is compact, this assumption does not restrict preferences at all in consumer i's attainable consumption set. Assumption A.2 ensures that some (positive) amount of the public good will be produced at any Nash equilibrium. It is not a necessary assumption but technically avoids considerations of boundary cases.

Theorem 2: Under the general assumptions on the utility function u^i and Assumptions A.1 and A.2, the game F^Q has a Nash Equilibrium $\hat{m}$.

Proof: We outline a proof of this theorem through a series of lemmas.

Lemma 1: If $\psi^i: M^{I-1} \to M$ is defined by: $m^i = \psi^i(m^{)i(})$ is the solution of equation (3.5) for every $m^{)i(} \in M^{I-1}$ and $\psi = (\psi^1, \ldots, \psi^I): M^I \to M^I$, then ψ is a well-defined continuous function and a fixed point of ψ is a Nash equilibrium of the game F^Q.

Proof: Straightforward.

Now, to use the Brouwer Fixed Point Theorem to prove ψ has a fixed point we need to find a compact convex set $\bar{M} \subset M^I$ such that ψ maps $\bar{M}$ into $\bar{M}$. There does not appear to be an obvious natural set[9/] and thus we resort to the standard trick of general equilibriums analysis of bounding M artificially and then relaxing the bound. Thus for every integer $T > 0$, let M_T be the (closed) interval $[-T, IT]$, $M_T \equiv [-T, IT]$, and let $\psi_T^i(\cdot)$ be i's reaction function when restricted to choose messages from M_T: i.e.

$$\psi_T^i(m^{)i(}) = m^i \quad \text{solves} \quad \max_{m^i \in M_T} v^i(m/m^i) \quad .$$

Lemma 2: $\psi_T^i(\cdot)$ is a continuous function from $(M_T)^{I-1}$ into M_T.

Proof: Straightforward.

Thus, $\psi_T = (\psi_T^1, \ldots, \psi_T^I)$ is a continuous function from $(M_T)^I$ into $(M_T)^I$ and as $(M_T)^I$ is compact and convex, $\psi_T(\cdot)$ has a fixed point m_T by the Brouwer Fixed Point Theorem.

Lemma 3: If for some value of $T > 0$, $-T < m_T^i < IT$ for every i (where m_T is a fixed point of $\psi_T(\cdot)$), then $m_T \equiv \hat{m}$ is a Nash Equilibrium for the game F^Q.

Proof: Immediate, since if the constraints on the messages m_T^i are not binding, then $m_T^i = \psi(m_T^i)$; i.e. m_T^i is a fixed point of $\psi(\cdot)$ (as well as $\psi_T(\cdot)$) and thus, by Lemma 1, is a N.E.

Thus, we need to find some T^* sufficiently large to guarantee that $-T^* < m_{T^*}^i < IT^*$ for all i.

Lemma 4: Under Assumption A.2, for every $T \geq 1$, $y(m_T) > 0$ at the fixed point m_T of $\psi^T(\cdot)$.

Proof: Suppose not; $y(m_T) \leq 0$ for some $T \geq 1$. By Assumption A.2, quasi-concavity of $u^i(\cdot)$, and since $\Sigma_i x^i(m_T) = \Sigma_i w^i - \Sigma_i C^i(m_T)$ $= \Sigma_i w^i - y(m_T) \geq \Sigma w^i$,

$$\Sigma_i \pi^i(x^i(m_T), y(m_T)) > 1 = \Sigma_i \frac{\partial C^i(m_T)}{\partial m^i} \quad .$$

Thus, for some i_0,

$$\pi^{i_0}(x^{i_0}(m_T), y(m_T)) > \frac{\partial C^{i_0}(m_T)}{\partial m^{i_0}}$$

which implies the upper constraint is binding for i_0 and thus $m^{i_0} = IT$. But, then, $y(m_T) = IT + \Sigma_{h \neq i_0} m_T^h \geq IT - (I-1)T = T > 0$. Contradiction.

Lemma 5: Under Assumption A.1, for every $T \geq 1/\gamma$, $m_T^i > -T$ for all i.

Proof: Suppose not; $m_T^i = -T$ some i and T. Then

$$\pi^i \Big|_{m_T} \leq \frac{\partial C^i}{\partial m^i} = \alpha^i + \gamma(m_T^i - \frac{1}{I} y(m_T)) \leq \alpha^i - \gamma T$$

as $y(m_T) > 0$ by Lemma 4. Thus $\pi^i \big|_{m_T} \leq \alpha^i - \gamma T < 0$ for $T \geq 1/\gamma$ contradicting the strict monotonicity of $u^i(\cdot,\cdot)$.

Lemma 6: Under Assumptions A.1 and A.2, there exists some $T^* \geq 1/\gamma$ such that $m_{T^*}^i < IT^*$ for every i.

Proof: Suppose not; then there is some i and a sequence $T_t \to \infty$ as $t \to \infty$ such that $m_t^i \equiv m_{T_t}^i = IT_t$ all t. But then

$$\pi^i \Big|_{m_t} \geq \alpha^i + \gamma(m_t^i - \frac{1}{I} y(m_t)) = \alpha^i + \gamma(IK_t - \frac{1}{I} y(m_t)) \geq \alpha^i$$

as $I^2 T_t \geq y(m_t) \geq T_t$ all t.

Thus, as $t \nearrow \infty$, $T_t \nearrow +\infty$ and $y(m_t) \to +\infty$ and $\pi^i \big|_{m_t} \geq \alpha^i > 0$ all t which would contradict Assumption A.1 if $x^i(m_t) \leq \bar{x}^i$ for some $\bar{x}^i$ all t. Since

$$x^i(m_t) = w^i - \alpha^i y(m_t) - \frac{\gamma}{2}[\frac{I-1}{I}(m_t^i - \mu_t^i)^2] + \frac{\gamma}{2} \sigma_t^{i^2}$$

it suffices to show that $\sigma_t^{i^2}$ is bounded above.

Now at the fixed point m_t, since $\pi^h \geq 0$ and $m_t^h > -T_t$ by Lemma 5, for every h, either

$$\pi^h\Big|_{m_t} \geq \alpha^h + \gamma[m_t^h - \tfrac{1}{I}y(m_t)] \quad , \text{ in which case } \quad m_t^h = IT_t$$

or $$\pi^h\Big|_{m_t} = \alpha^h + \gamma[m_t^h - \tfrac{1}{I}y(m_t)] \geq 0 \quad , \quad \text{in which case } \quad IT_t \geq m_t^h \geq \tfrac{1}{I}y(m_t) - \tfrac{1}{\gamma}\alpha^h$$

Thus in either case $IT_t \geq m_t^h \geq \frac{1}{I}y(m_t) - \frac{1}{\gamma}\alpha^h \geq \frac{1}{I}y(m_t) - \frac{1}{\gamma}$ for all h, and thus $IT_t \geq \mu_t^i \geq \frac{1}{I}y(m_t) - \frac{1}{\gamma}$.

Hence

$$(IT_t - \tfrac{1}{I}y(m_t) + \tfrac{1}{\gamma}) \geq |m_t^h - \mu_t^h| \geq 0 \quad \text{all } h \text{ which implies}$$

$$\sigma_t^{i^2} = \frac{1}{I-2}\Sigma_h(m_t^h - m_t^h)^2 \leq \frac{I-1}{I-2}(IT_t - \tfrac{1}{I}y(m_t) + \tfrac{1}{\gamma})^2 \quad .$$

But $y(m_t) = m_t^i + \Sigma_{h\neq i}m_t^h \geq IT_t + \frac{I-1}{I}y(m_t) - \frac{1}{\gamma}(\Sigma_{h\neq i}\alpha^h)$ which implies

$$0 \leq IT_t - \tfrac{1}{I}y(m_t) \leq \tfrac{1}{\gamma}\Sigma_{h\neq i}\alpha^h \leq \tfrac{1}{\gamma} \quad \text{and} \quad (IT_t - \tfrac{1}{I}y(m_t) + \tfrac{1}{\gamma})^2 \leq \frac{4}{\gamma^2}$$

and

$$\sigma_t^{i^2} \leq \frac{I-1}{I-2}\cdot\frac{4}{\gamma^2} \leq \frac{8}{\gamma^2} \quad \text{as} \quad n \geq 3 \quad .$$

Lemma 6 thus completes the proof of Theorem 2. Q.E.D.

Although we have proved the existence of a Nash equilibrium for the game F^Q, in order to conclude the existence of a competitive equilibrium under G^Q we need to ensure that the allocation $(\langle x^i(m)\rangle, y(m))$ defined by the Nash equilibrium is individually feasible for every i. The assumptions introduced thus far are not sufficient unless private good consumption is unbounded from below (i.e. $X^i = \mathbb{R} \times \mathbb{R}_+$).

The crux of the difficulty lies in the fact that the tax rules $C^i(\cdot)$ of the government are potentially confiscatory of a consumer's total endowment. For example, suppose all consumers but one are identical and have such strong preferences for the public good that they are willing to spend any positive wealth on the public good while the remaining consumer, 1, is indifferent to the public good and thus always attempts to minimize his tax. However, since he is atypical, in addition to paying his fixed share α^1 of whatever quantity is purchased, he must also pay for the deviation of his message from the others' mean: $[\gamma(I-1)/2I](m^1 - \mu^1)^2$. Each one of the similar consumers thus will have his tax <u>reduced</u> from his fixed share α^i so that in aggregate the similar consumers pay their fixed shares $\Sigma_{i\neq 1}\alpha^i$ <u>less</u> the amount $[\gamma(I-1)/2I](m^1 - \mu^1)^2$ received from the atypical consumer. Now suppose each consumer's fixed share α^h is set equal to his relative wealth $w^h/\Sigma_i w^i$. (If not, it is easy to construct examples to bankrupt any consumer with a greater fixed share.) Then, at any Nash equilibrium, $\hat{m}$, the similar consumers are spending all their wealth on the public good and the atypical consumer 1 is minimizing his cost. Thus

$$\Sigma_{i\neq 1}C^i(\hat{m}) = \Sigma_{i\neq 1}w^i + \frac{\gamma(I-1)}{2I}(\hat{m}^1 - \hat{\mu}^1)^2 = \Sigma_{i\neq 1}\alpha^i \cdot y(\hat{m}) - \frac{\gamma(I-1)}{2I}(\hat{m}^1 - \hat{\mu}^1)^2$$

which implies that

$$y(\hat{m}) = \Sigma_i w^i + \frac{\frac{\gamma(I-1)}{I}(\hat{m}^1 - \hat{\mu}^1)^2}{\Sigma_{i\neq 1}\alpha^i} > \Sigma_i w^i \quad \text{since} \quad \hat{m}^1 \neq \hat{\mu}^1$$

as 1 is not similar to the other consumers. But as $\Sigma_i C^i(\hat{m}) = y(\hat{m}) > \Sigma_i w^i$ and $x^i(\hat{m}) = w^i - C^i(\hat{m}) = 0$ for all $i \neq 1$,

$$x^1(\hat{m}) = \Sigma_i x^i(\hat{m}) = \Sigma_i w^i - \Sigma_i C^i(\hat{m}) = \Sigma_i w^i - y(\hat{m}) < 0 \quad .$$

Since 1 is cost minimizing at $\hat{m}^1$, yet $x^1(\hat{m}) < 0$, consumer 1 is bankrupt at any Nash equilibrium.

More generally, this type of bankruptcy arises if (i) there exist sufficient diversity in the preferences for public goods, and simultaneously, (ii) aggregate demand at the fixed cost share prices α^i equal to relative wealth is close to the maximum feasible output of the public good for the economy. When these two conditions exist, there may not be enough private good left over after producing the demanded high quantity of public good to serve as the medium of transfers to compensate for the diversity of tastes.

To avoid the bankruptcy problem, and thus prove a competitive equilibrium exists, we rule out preferences leading to near total public good production. Specifically, consider the two assumptions.

Assumption A.3: For any i, $\pi^i(0,y) < w^i/\Sigma_h w^h$ for all $y > 0$.

Assumption A.4:[10/] $\Sigma_i \pi^i(x^i,y) < 1$ for all $(\langle x^i \rangle_i, y)$ such that

$$y = \Sigma_h w^h - \Sigma_h x^h \quad \text{and} \quad \Sigma_h x^h \leq \frac{1}{\gamma} \quad .$$

Assumption A.3 is a technical assumption made to ensure no consumer would ever choose to spend his entire income on the public good if charged $\alpha^i = w^i/\Sigma_h w^h$ per unit at the margin. Such boundary cases could be allowed

if the smoothness of u^i were not assumed. If $\pi^i(x^i,y)$ is discontinuous at $x^i = 0$ then it is sufficient to assume Assumption A.3 holds at all $x^i < 0$, an assumption that places no restriction on preferences in the feasible set.

Assumption A.4 is the substantive assumption needed to ensure that nearly all the private good endowment will not be used to produce the public good, regardless of which attainable private good distribution is realized. The number $1/\gamma$ is the minimum amount of private good available at any efficient allocation.[11/]

Theorem 3: Under the general assumptions on the utility functions u^i and Assumptions A.1–A.4, if $\alpha^i = w^i/\Sigma_h w^h$ then at any Nash equilibrium $\hat{m}$ of F^Q, the allocation $(\langle x^i(\hat{m})\rangle, y(\hat{m}))$ is individually feasible.

Proof: By Lemma 4, $y(\hat{m}) > 0$. Thus, we need only establish that $x^i(\hat{m}) \geq 0$ for all i.

Claim: $\hat{y} = y(\hat{m}) \leq \Sigma_h w^h - \frac{1}{\gamma}$.

Proof: At $\hat{m}$, $\Sigma_i \pi^i(\hat{x}^i, y) = 1$ and $\hat{y} = \Sigma_h w^h - \Sigma_h \hat{x}^h$. Thus, by Assumption 4 and the quasi-concavity of u^i,

$$\hat{y} \leq \Sigma_h w^h - \frac{1}{\gamma} \quad .$$

Now

$$x^i(\hat{m}) = w^i - \alpha^i \hat{y} - \frac{\gamma}{2}\left(\frac{I-1}{I}\right)(\hat{m}^i - \hat{\mu}^i)^2 + \frac{\gamma}{2}\hat{\sigma}^{i\,2} \geq w^i - \alpha^i \hat{y} - \frac{\gamma}{2}\left(\frac{I-1}{I}\right)(\hat{m}^i - \hat{\mu}^i)^2 \quad .$$

Since at the N.E. $\hat{m}$, $(\hat{m}^i - \hat{\mu}^i) = \frac{I}{\gamma(I-1)}(\hat{\pi}^i - \alpha^i)$ where $\hat{\pi}^i = \pi^i|_{\hat{m}}$

$$x^i(\hat{m}) \geq w^i - \alpha^i y - \frac{1}{2\gamma}\left(\frac{I}{I-1}\right)(\hat{\pi}^i - \alpha^i)^2 \quad \text{and as} \quad \alpha^i = w^i/\Sigma_h w^h$$

$$\hat{y} \leq \Sigma_h w^h - \frac{1}{\gamma} \quad , \quad \text{and} \quad 1 \geq \frac{1}{2}\left(\frac{I}{I-1}\right)\alpha^i \quad ,$$

$$x^i(\hat{m}) \geq \frac{\alpha^i}{\gamma} - \frac{1}{2\gamma}\left(\frac{I}{I-1}\right)(\hat{\pi}^i - \alpha^i)^2 \geq \frac{1}{2\gamma}\left(\frac{I}{I-1}\right)\hat{\pi}^i(2\alpha^i - \hat{\pi}^i) \quad .$$

If $x^i(\hat{m}) \leq 0$, then $\hat{\pi}^i < \alpha^i$ by quasi-concavity of u^i and Assumption A.3 which implies that $x^i(\hat{m}) > 0$ which is a contradiction. Thus, $x^i(\hat{m}) \geq 0$ as was to be shown.

Corollary: Under the assumptions of Theorem 3, there exists a competitive equilibrium under G^Q. Furthermore, under these assumptions every competitive equilibrium is Pareto optimal.

Proof: Theorem 2 and Theorem 3 imply there exists a N.E. $\hat{m}$ at which the allocation $(\langle x^i(\hat{m})\rangle, y(\hat{m}))$ is individually feasible. Thus $\hat{m}$ defines a competitive equilibrium under G^Q Furthermore, Theorem 3 implies every N.E. $\hat{m}$ yields individually feasible allocations. Hence, the set of N.E. and competitive equilibrium coincide and every competitive equilibrium is interior. By Theorem 1, then, every competitive equilibrium is Pareto Optimal.

Although the assumption of Theorem 3 that the proportional cost shares, α^i, equal proportional wealth, $w^i/\Sigma_h w^h$, does not seem to be an unimplementable requirement, it is interesting to know what would happen

if the parameters α^i were not dependent on the particular data of the economy, except perhaps, its size: (e.g., if $\alpha^i = 1/I$). It can be shown that Theorem 3 remains valid without the requirement that $\alpha^i = w^i/\Sigma_h w^h$ if Assumption A.3 is replaced by:

Assumption A.3': For any i, $\pi^i(0,y) < \alpha^i$ for all $y > 0$,

and Assumption 4 is replaced by:

Assumption A.4': $\Sigma_i \pi^i(x^i,y) < 1$ for all $(\langle x^i\rangle_i, y)$ such that $y = \Sigma_i w^i - \Sigma_i x^i$ and $\Sigma_i x^i \leq I/(2\gamma(I-1)) + \operatorname{Max}_i \{\Sigma_h w^h - (w^i/\alpha^i)\}$.

4. Existence of a General Competitive Equilibrium under the Quadratic Government

4.1 The Assumptions

For the general economy defined in Section 2 $E = \{\langle X^i, \succsim_i, \omega^i\rangle, \langle Z^j\rangle, \langle\theta^{ij}\rangle\}$ we assume the following standard conditions of general equilibrium analysis:[12]

Standard Assumptions: E satisfies, for every i and j

(a) $X^i = x^i \times \mathbb{R}^K_+$; $x^i \subseteq \mathbb{R}^L$, x^i is closed, convex, and has a lower bound for $\leq$.

(b.1) for every $(x^i,y) \in X^i$, there exists an $x^{i'}$ such that $(x^{i'},y) \in X^i$ and $(x^{i'},y) \succ_i (x^i,y)$. (non-satiation in private goods)

(b.2) for every $(\bar{x}^i,\bar{y}) \in X^i$, the sets $\{(x^i,y) \in X^i \mid (x^i,y) \succsim_i (\bar{x}^i,\bar{y})\}$ and $\{(x^i,y) \in X^i \mid (x^i,y) \precsim_i (\bar{x}^i,\bar{y})\}$ are closed. (continuity of preferences)

(b.3) for every (x^i,y) and $(\bar{x}^i,\bar{y}) \in X^i$, if $(x^i,y) \succ_i (\bar{x}^i,\bar{y})$ and $0 < \lambda < 1$ then $\lambda(x^i,y) + (1-\lambda)(\bar{x}^i,\bar{y}) \succ_i (\bar{x}^i,\bar{y})$. (convexity of preferences)

(c) $\omega^i \in \text{int } X^i$. (feasibility of the initial endowment)

(d.1) $0 \in Z^j$. (possibility of inaction)

(d.2) Z is closed and convex where $Z = \Sigma_j Z^j$ is the aggregate production set of E.

(d.3) $Z \cap (-Z) \subset \{0\}$. (irreversibility of production)

(d.4) $Z \supset (-\Omega)$ where $\Omega \equiv \mathbb{R}_+^{L+K}$. (free disposal)

As for the simple neo-classical model of Section 3, the standard assumptions alone are not sufficient to prove the existence of a competitive equilibrium under the government G^Q, because of the possibility some consumers may be driven into bankruptcy when all agents maximize preferences or profits. Bankruptcy may result under the tax rules of G^Q either if all private goods prices are driven to zero or if the demand for public goods is too great.[13] To rule out these possibilities we make two additional assumptions. First, we assume technology would permit the production of more of every public good than is possible at any attainable state if resources were large enough. Second, we assume that at a feasible allocation, if every consumer is too close to the boundary of the private goods portion of his consumption set or, put another way, if public goods output could not be increased some incremental amount without requiring more private good inputs than would be available, then all consumers would prefer some feasible allocation at which the amount of public goods were smaller.

Formally, let A denote the set of attainable states for the economy E:

(4.1) $$A \equiv \{a = (\langle x^i\rangle, y, \langle z^j\rangle) \mid (x^i, y) \in X^i,\ z^j \in Z^j,\ (\Sigma_i(x^i - \omega^i), y) = \Sigma_j z^j\}$$

Note that A depends only on the consumption sets $X^i = X^i \times \mathbb{R}_+^K$, the production sets Z^j, and the aggregate endowment of private goods, $\Sigma_i \omega^i = \omega$. Let A_y denote the set of attainable public goods bundles:

(4.2) $$A_y \equiv \{y \in \mathbb{R}_+^K \mid \text{there exists } a' \in A \text{ with } y' = y\} .$$

Our first additional assumption is:

<u>Assumption (d.5)</u>: Given any $y \in A_y$, $(z_x, y + c\underline{1}) \in Z$ for some $c > 0$ and some $z_x \in \mathbb{R}^L$, where $\underline{1} \equiv (1,\ldots,1) \in \mathbb{R}^K$.

To define our other assumption, let H denote those public goods bundles in A_y that are bounded away from the upper boundary by the amount $1/\gamma$ where γ is the parameter on the quadratic terms of the tax rules for the Quadratic government G^Q:

(4.3) $$H \equiv \{y \in A_y \mid y + \tfrac{1}{\gamma}\underline{1} \in A_y\}$$

Our second additional assumption is:

<u>Assumption (e)</u>: If $a = (\langle x^i\rangle, y, \langle z^j\rangle) \in A$ and $y \notin H$, then there exists some $a' = (\langle x^{i\prime}\rangle, y', \langle z^{j\prime}\rangle) \in A$ with $y' \in H$ such that $(x^{i\prime}, y') \succ_i (x^i, y)$ for all i.

Note that Assumption (e) can be satisfied only if H is not empty. Thus, minimally the amount $1/\gamma > 0$ of every public good must be compatible with attainability. Of course, the larger is γ, the less restrictive is the requirement. However since $A_y = \{0\}$ for a private-goods only economy, such economies are not covered by the existence theorem below. But if it is assumed that public goods are never undesireable at zero levels, then the zero point, $y = 0$, may be adjoined to H and the theorem will cover the private-goods only economy also.

A weaker but more complicated assumption may be substituted for Assumption (e). Let

$$P = \{a \in A \mid \nexists\ a' \in A \text{ such that } (x^{i'}, y') \succ_i (x^i, y) \text{ for all } i\}.$$

P is the set of (weak) Pareto-optimal allocations. Under Assumptions (a)-(d.5), if $a \in P$, then there exists a support (price) vector s in $S \equiv \{s \in \mathbb{R}^{i+k} \mid \|s\| = \Sigma_\ell p_\ell + \Sigma_k q_k = 1\}$ such that $s \cdot z \geq s \cdot z'$ for all $z' \in P(z) \equiv \{z \in \mathbb{R}^{L+K} \mid z = \Sigma_j z^j$ for some $a \in A\}$. Let $S(a)$ be the set of all such support prices. These prices are not necessarily equilibrium prices as they depend only on the technology and the Pareto-optimal allocation a. The weaker assumption is.

<u>Assumption (e')</u>: If $a \in P$, then for all $s \in S(a)$,

$$q \cdot y < \Sigma_h [w^h(s) - \min_{x^h \in X^h} p \cdot x^h] - \frac{1}{\gamma} \quad .$$

For the example of Section 3, both Assumptions (e) and (e') are equivalent to Assumption A.4. It can also be shown that Assumption (e) implies Assumption (e').

4.2 <u>The Existence Theorem: Statement</u>

The theorem we prove is:

<u>Theorem 4.1</u>: The economy with public goods E has a competitive equilibrium under the quadratic government G^Q under Assumptions (a), (b.1)-(b.3), (c), (d.1)-(d.5), and (e) when the parameters α^i of the tax rules for G^Q are specified by:

$$\alpha^i(s) = \frac{w^i(s) - \min_{x^i \in X^i} p \cdot x^i}{\Sigma_h [w^h(s) - \min_{x^h \in X^h} p \cdot x^h]} \tag{4.4}$$

(where $w^i(s) = p \cdot \omega^i + \Sigma_j \theta^{ij} \pi^j(s)$ is consumer i's wealth at prices s; see Definition 2.4).

<u>Remark</u>: As in Section 3, restriction (4.4) on the parameters α^i may be removed if Assumption (e) or (e') is suitably strengthened. For example, consider:

<u>Assumption (e")</u>: If $a \in P$ then, for all $s \in S(a)$ and all i,

$$\alpha^i q \cdot [y + \frac{1}{\gamma}\underline{1}] < w^i(s) - \min_{x^i \in X^i} p \cdot x^i \quad .$$

Theorem 4.1 in valid if (e) is replaced by (e") and restrictions (4.4) are eliminated. (Also, in Assumption (e') and (e"), the scalar $(1/\gamma)$ can be replaced by the smaller scalar $[I/2\gamma(I - 1)]$.)

To see how Assumption (e") relates to Assumption (a.4') of Section 3, suppose, in the model of Section 3, that $a \in P$. Then, by (A.4'),

$$\Sigma_i x^i > \frac{I}{2\gamma(I-1)} + \text{Max}\,\{\Sigma_h w^h - \frac{w^i}{\alpha^i}\}$$

or, using the fact that $y = \Sigma_h w^h - \Sigma_i x^i$ in that model,

$$\alpha^i(y + \frac{I}{2\gamma(I-1)}) < w^i \quad \text{for every} \quad i \quad .$$

Furthermore, since $p = q = 1$ and $\underset{x^i \in X^i}{\text{Min}}\ p \cdot x^i = 0$ as $X^i = \mathbb{R}_+$, assumption (e") is also satisfied (after replacing $1/\gamma$ with $I/2\gamma(I-1)$).

Although the importance of prices in Assumptions (e') and (e") may seem strange, it should be noted that the tax rules $C^i(\cdot)$ specify payment only in the unit of account and the purpose of Assumption (e') or (e") is to ensure that there is a sufficient amount of the unit of account to carry out the required transfers. Assumption (e) is stronger (than (e')) since it requires sufficient amount of every private commodity to be available to carry out the transfers if the tax rules required payment in any particular commodity.

4.3 Proof of Existence Theorem

We present a numbered outline of the proof of Theorem 4.1 which follows in many details Debreu's existence proof for a private-goods-only economy. Thus, where possible, we refer to the relevant paragraphs of Debreu's proof in [1959].

(1) The set of attainable states A defined in (4.1) is non-empty, convex, and compact.

Proof: Same as in Debreu [1959].

(2) Compactify the message space M as in Section 3 for the simple economy: For every $t \geq 1$, let

$$M_t \equiv \{m^i \in M \mid -t \leq m_k^i \leq It, \text{ all } k\} \quad .$$

Clearly M_t is non-empty, convex, and compact for all t.

(3) Compactify the economy $\mathcal{E}$: Let $\bar{X}^i$ and $\bar{Z}^j$ denote the projections of the attainable set A onto X^i and Z^j respectively. By (1), $\bar{X}^i$ and $\bar{Z}^j$ are compact and convex.

For any number $n \in \mathbb{R}$, let $B^N(n)$ denote the N-dimensional cube centered at the origin with edges of length $2n$; i.e.

$$B^N(n) \equiv \{g \in \mathbb{R}^N \big| |g_i| \leq n \quad \text{all} \quad i = 1,\dots,N\} \quad .$$

Given any $t \geq 1$ let $n(t) \in \mathbb{R}$ be sufficiently large so that

(i) $\bar{X}^i$, $\bar{Z}^j$ are contained in $B^{L+K}(n(t))$, all i, j

(ii) $Y_t \equiv y(M_t^I) \equiv \{y \in \mathbb{R}^K | y = y(m) = \Sigma_h m^h,\ m \in M_t^I\} \subset B^K(n(t))$ and

(iii) $B^L(n(t))$ contains the lower bound of X^i, all i (see (a)).

Define $X_t^i \equiv X^i \cap B^L(n(t))$, $\mathcal{X}_t^i \equiv \mathcal{X}^i \cap B^{L+K}(n(t))$, $Z_t^j \equiv Z^j \cap B^{L+K}(n(t))$.
Clearly these spaces are non-empty, convex, and compact for all t.

(4) Define compactified supply correspondences $\phi_t^j(\cdot)$ and profit functions $\pi_t^j(\cdot)$ as the restrictions of $\phi^j(\cdot)$ and $\pi^j(\cdot)$ respectively, to Z_t^j. As in Debreu ([1959], p. 86, 4 & 5), $\pi_t^j(\cdot)$ is continuous and $\phi_t^j(\cdot)$ is non-empty and convex valued and upper semi-continuous (≡ u.s.c., hereafter) for every $s \in \mathbb{R}^{L+K}$.

<u>(A) Discussion</u>: It is not possible at this point in our proof to follow Debreu and compactify the consumer's decision correspondence $\delta^i(\cdot)$ and proceed to the compactified excess demand correspondence. As we have noted above, under the tax rules of G^Q, a consumer's budget set $\beta^i(m^{)i(},s;\, w^i(s))$ (see Definitions 2.3 and 2.4) <u>may</u> be empty for some $m^{)i(}$ and s; i.e. consumer i may be bankrupt. Although Assumptions (d.5) and (e) are sufficient to prove no consumer is bankrupt at a "fixed point" which we show defines an equilibrium, to prove the "fixed point" exists, we need a <u>non-empty</u>, convex valued, and u.s.c. decision correspondence for each consumer.

Thus, we define a pseudo-decision correspondence which agrees with the decision correspondence $\delta^i(\cdot)$ (see Definition 2.3) if the consumer is not bankrupt, but allows him to choose cost-minimizing consumption and message pairs (x^i,m^i) if he is bankrupt. However, for technical reasons, whenever strict cost-minimization would eliminate the bankruptcy (this can happen only if $\Sigma_{h\neq i} m_k^h + \underline{m}_k^i = y_k(m) < 0$ for some k where $\underline{m}^i$ minimizes $C^i(m,s)$ over M) we allow him to cost minimize only to the brink to solvency.

(5) Therefore, let $\tilde{\delta}^i(m^{)i(},s;\, w^i)$ denote consumer i's <u>pseudo-decision</u> correspondence and be defined by:

$$\tilde{\delta}^i(m^{)i(},s;\, w^i) \equiv \begin{cases} \delta^i(m^{)i(},s;\, w^i) & \text{if} \quad d^i(m^{)i(},s) < w^i \\ \beta^i(m^{)i(},s;\, w^i) & \text{if} \quad d^i(m^{)i(},s) = w^i \\ \xi^i(m^{)i(},s;\, w^i) & \text{if} \quad d^i(m^{)i(},s) > w^i \end{cases}$$

where $d^i(m^{)i(},s) \equiv \underset{x^i\in X^i}{\mathrm{Min}}\ p\cdot x^i + \underset{\substack{m^i\in M \\ \text{s.t. } y(m)\geq 0}}{\mathrm{Min}}\ C^i(m/m^i,\, s)$ is the minimum cost to get into his consumption set, and $\xi^i(\cdot)$ is defined by:

$$\xi^i(m^{)i(},s;\, w^i) \equiv \Big\{(\bar{x}^i,\bar{m}^i) \in X^i \times M \,\big|\, \bar{x}^i \text{ minimizes } p\cdot x^i \text{ over } X^i,$$

and either

(a) $\bar{m}^i = \underline{m}^i(m^{)i(},s) = \underline{m}^i$ minimizes $C^i(m,s)$ over M subject to

$$m_k^i \geq \begin{Bmatrix} -(I-1)\mu_k^i \\ \\ \mu_k^i \end{Bmatrix} \quad \text{as} \quad \mu_k^i \begin{Bmatrix} \geq \\ < \end{Bmatrix} 0 \ , \quad \text{every } k \ ;$$

if $C^i(m/\underline{m}^i,s) > w^i - \underset{x^i\in X^i}{\mathrm{Min}}\ p\cdot x^i$, or

(b) $\bar{m}^i$ maximizes $q\cdot \mathrm{Min}\ \{0,y(m)\}$ subject to

(i) $$m_k^i \geq \begin{Bmatrix} -(I-1)\mu_k^i \\ \\ \mu_k^i \end{Bmatrix} \quad \text{as} \quad \mu_k^i \begin{Bmatrix} \geq \\ < \end{Bmatrix} 0 \ , \quad \text{every } k \ ,$$

$$\underline{\text{and}}\ \text{(ii)}\quad C^i(m,s) \leq w^i - \underset{x^i \in X^i}{\text{Min}}\ p \cdot x^i$$

$$\text{if}\quad C^i(m/\underline{m}^i;\ s) \leq w^i - \underset{x^i \in X^i}{\text{Min}}\ p \cdot x^i \Big\}.$$

(B) Discussion: In the definition above the pseudo-decision (≡ p.-decision, hereafter) correspondence, the consumer's wealth w^i is an exogenous variable. Typically, for private ownership general equilibrium models, income is endogenously determined as the value of the initial endowment, plus the shares of firms' profits: $w^i(s) \equiv p \cdot \omega^i + \Sigma_j \theta^{ij}\pi^j(s)$.

However, in our model, since the p.-decision correspondence allows a consumer's decision (x^i, m^i) to violate the budget constraint under some circumstances, if w^i is set equal to $w^i(s)$ a situation may arise in which the value of aggregate excess demand is strictly positive, i.e. Walras' Law may be violated. Since our proof requires us to show Walras' Law holds, we must modify the income determination process. Loosely speaking, in the presence of any bankruptcy, we invoke a redistribution mechanism. All non-bankrupt consumers are charged in proportion to their solvency level to cover the deficits of the bankrupt consumers.

(6) Define consumer i's degree of solvency (if positive) or bankruptcy (if negative) by:

$$b^i(m)^i(,s) \equiv w^i(s) - d^i(m)^i(,s) \quad \text{where} \quad d^i(\cdot) \quad \text{is defined above at (5)}$$

Let $r^i(m,s)$ denote i's assessment for bankruptcy (bankruptcy tax) and be defined by

$$r^i(m,s) \equiv \begin{cases} 0 \quad \text{if}\ b^i(m)^i(,s) \leq 0 \ ;\ \text{i.e. if}\ i\ \text{is bankrupt} \\ \text{Min}\left\{ \dfrac{b^i(m)^i(,s)}{\sum\limits_{\substack{h \\ b^h>0}} b^h(m)^h(,s)} \times \Big| \sum\limits_{\substack{h \\ b^h<0}} b^h(m)^h(,s) \Big|,\ b^i(m)^i(,s) \right\} \\ \quad \text{if}\ b^i(m)^i(,s) > 0 \ . \end{cases}$$

Note that when all consumers are solvent; i.e. $b^i \geq 0$, then $r^i = 0$, i.e. bankruptcy taxes are zero. Note also that i's bankruptcy tax will never bankrupt him; i.e. $b^i(m)^i(,s) > 0$ implies after tax solvency $b^i(m)^i(,s) - r^i(m,s) = (w^i(s) - r^i(m,s)) - d^i(m)^i(,s) \geq 0$.

Now, the consumer's p.-decision correspondence (with endogenous income (wealth) determination) is defined simply by:

$$\tilde{\delta}^i(m,s) \equiv \delta^i(m)^i(,s;\ w^i(s) - r^i(m,s)) \ .$$

(7) We now compactify the consumer's p.-decision correspondence $\tilde{\delta}(\cdot)$ by substituting the compactified sets M_t, X_t^i, x_t^i, and Z_t^j everywhere in the definition of all elements of the model for the original sets M, X^i, x^i, and Z^j. This process will define the functions or correspondences $w_t^i(\cdot)$, $\alpha_t^i(\cdot)$, $C_t^i(\cdot)$, $\beta_t^i(\cdot)$, $\delta_t^i(\cdot)$, $d_t^i(\cdot)$, $\xi_t^i(\cdot)$, $\hat{\delta}_t^i(\cdot)$, $b_t^i(\cdot)$, $r_t^i(\cdot)$, and finally, $\tilde{\delta}_t^i(\cdot)$, the compactified p.-decision correspondence. Note that the tax rules $C_t^i(\cdot)$ were also compactified in the process.

The compactified p.-decision correspondence $\tilde{\delta}_t^i(\cdot)$ can be shown to have the required properties:

<u>Lemma 1</u>: $\tilde{\delta}_t^i(\cdot)$ is non-empty, convex valued and u.s.c. on $M_t^I \times \overset{\circ}{S}$ where $\overset{\circ}{S} = \{s = (p,q) \in \mathbb{R}_+^{L+K} | \Sigma_\ell p_\ell + \Sigma_k q_k \equiv \|s\| = 1, \|p\| > 0\}$ is the price simplex open at $\|p\| = 0$.

<u>Proof</u>: Straightforward, but tediously detailed.

Note that for the definition of $\alpha^i(\cdot)$ given in (4.4), if $\|p\| = 0$, $\alpha^i(s)$ may not be well-defined.

(8) Let the space of excess demands for the compactified economy be defined by:

$$E_t \equiv \{e \in \mathbb{R}^{L+K} | e = (\Sigma_i(w^i - \omega^i), y) - \Sigma_j z^j,\ x^i \in X_t^i,\ y \in Y_t,\ z^j \in Z_t^j\}$$

and let S_ν denote the closed subset of prices:

$$S_\nu \equiv \{s \in \overset{\circ}{S} | \|p\| \geq \nu\} \quad , \quad \text{for every } 1 > \nu > 0 \ .$$

Clearly, the sets E_t and S_ν are non-empty, compact, and convex since X_t^i, $Y_t \equiv y(M_t^I)$, and Z_t^j are.

Now define the "maximal valuation of excess demand" correspondence $\eta_{t\nu}: E_t \to S_\nu$ for every t, ν by:

$$\eta_{t\nu}(e) = \{s' \in S_\nu \,|\, s' \cdot e \geq s \cdot e \quad \text{for all } s \in S_\nu\} \ .$$

As in Debreu ([1959], (1) of (5.6)), $\eta_{t\nu}(\cdot)$ is non-empty, convex, and u.s.c. at all $e \in E_t$ for all $t > 1/\gamma$, $1 > \nu > 0$.

(9) Define the "fixed point" mapping $\rho_{t\nu}$: $E_t \times M_t^I \times S_\nu \to E_t \times M_t^I \times S_\nu$

$$\rho_{t\nu}(e,m,s) = \{(e',m',s') \in E_t \times M_t^I \times S_\nu | e' = (\Sigma_i(x^{i'} - \omega^i), y') - \Sigma_j z^{j'}$$
$$\text{for } y' = y(m'),\ (x^{i'}, m^{i'}) \in \tilde{\delta}_t^i(m,s),\ z^{j'} \in \phi_t^j(s),$$
$$s' \in \eta_{t\nu}(e)\} \ .$$

<u>Lemma 2</u>: The correspondence $\rho_{t\nu}(\cdot)$ is non-empty, convex valued and u.s.c. at every point in $E_t \times M_t^I \times S_\nu$ for every $t > 1/\gamma$, $1 > \nu > 0$

<u>Proof</u>: Straightforward.

Thus, by Kakutani's Fixed Point Theorem, for every $(t,\nu) > (1/\gamma, 0)$ $(\nu < 1)$, $\rho_{t\nu}(\cdot)$ has a fixed point; i.e. there exists

$$\varepsilon_{t\nu} = (\langle x_{t\nu}^i, m_{t\nu}^i \rangle, \langle z_{t\nu}^j \rangle, s_{t\nu}) \quad \text{such that}$$

(i) $(x_{t\nu}^i, m_{t\nu}^i) \in \tilde{\delta}_t^i(m_{t\nu}, s_{t\nu})$

(ii) $z_{t\nu}^j \in \phi_t^j(s_{t\nu})$

(iii) $s_{t\nu} \in \eta_{t\nu}(e_{t\nu})$ where $e_{t\nu} = (\Sigma_i(x_{t\nu}^i - \omega^i), y(m_{t\nu})) - \Sigma_j z_{t\nu}^j$.

<u>(C) Discussion</u>: It is not possible at this point in the proof to follow Debreu in one step and convert $\varepsilon_{t\nu}$ directly into an equilibrium (or, rather a pseudo-equilibrium for the compactified economy) by showing Walras' Law holds, thus that excess demand is non-positive, and hence that the free disposal assumption permits a modified production plan with no loss

in profits but which eliminates all excess supply. The difficulty is two-fold. First, in order to show Walras' Law holds even on the truncated price simplex S_ν it is necessary to show that the aggregate amount of bankruptcy is less than the aggregate solvency. Second to show Walras' Law holds on the entire simplex, we must consider the sequence of fixed points $\varepsilon_{t\nu}$ as ν goes to zero.

(10) <u>Lemma 3</u>: For every $t > 1/\gamma$, $1 > \nu > 0$, at the fixed point $\varepsilon_{t\nu}$

(a) $\Sigma_i b_t^i(m_{t\nu}^{)i(}, s_{t\nu}) > 0$

(b) $y(m_{t\nu}) \geq 0$

(c) $s_{t\nu} \cdot e_{t\nu} = 0$ and thus, $s \cdot e_{t\nu} \leq 0$ for all $s \in S_\nu$.

<u>Proof</u>: (a) Suppose to the contrary that $\Sigma_i b_t^i(m_{t\nu}^{)i(}, s_{t\nu}) \leq 0$. Then, by definition

$$r_t^i(m_{t\nu}, s_{t\nu}) = \begin{Bmatrix} 0 \\ b_t^i \end{Bmatrix} \text{ as } b_t^i \begin{Bmatrix} \leq \\ > \end{Bmatrix} 0 .$$

Thus, for every i,

$$d_t^i(m_{t\nu}^{)i(}, s_{t\nu}) \equiv w_t^i(s_{t\nu}) - b_t^i(m_{t\nu}^{)i(}, s_{t\nu}) \begin{Bmatrix} = \\ > \end{Bmatrix} w_t^i(s_{t\nu}) - r_t^i(m_{t\nu}, s_{t\nu})$$

as $b_t^i \begin{Bmatrix} \geq \\ < \end{Bmatrix} 0$.

Now, for each public good k, either (i) $y_k(m_{t\nu}) = \Sigma_h m_{kt\nu}^h > 0$ or (ii) $y_k(m_{t\nu}) \leq 0$.

(i) If $y_k(m_{t\nu}) > 0$, then $m_{kt\nu}^h = \mu_{kt\nu}^h - (\alpha^h(s_{t\nu})I/\gamma(I-1))q_{kt\nu}$ for all h.

(This follows since each consumer h is minimizing $C_t^h(m_{t\nu}/m^i, s_{t\nu})$ as $d_t^h(m_{t\nu}^{)h(}, s_{t\nu}) \geq w_t^h(s_{t\nu}) - r_t^h(m_{t\nu}, s_{t\nu})$. Also, when $y_k(m_{t\nu}) > 0$, none of the constraints on the cost minimizations are binding.)

But then, $\Sigma_h m_{kt\nu}^h = \Sigma_h \mu_{kt\nu}^h - (I/\gamma(I-1))q_{kt\nu} = \Sigma_h m_{kt\nu}^h - (I/\gamma(I-1))q_{kt\nu}$ implying that $q_{kt\nu} = 0$. Thus, $q_{kt\nu} \cdot y_k(m_{t\nu}) = 0$ if $y_k(m_{t\nu}) > 0$.

(ii) If $y_k(m_{t\nu}) \leq 0$, then $q_{kt\nu} y_k(m_{t\nu}) \leq 0$ as $q_{kt\nu} \geq 0$. all k.

Thus, in either event, since all consumers are cost minimizing,

$$(*) \qquad q_{t\nu} \cdot y(m_{t\nu}) = \Sigma_i C_t^i(m_{t\nu}, s_{t\nu}) \leq 0 .$$

Now by definition of $b_t^i(\cdot)$ and $w_t^i(\cdot)$,

$$(**) \qquad \Sigma_i b_t^i(m_{t\nu}^{)i(}, s_{t\nu}) = \Sigma_i (p_{t\nu} \cdot \omega^i - \min_{x^i \in X_t^i} p_{t\nu} \cdot x^i) + \Sigma_j \pi_t^j(s_{t\nu})$$

$$- \Sigma_i \min_{\substack{m^i \in M_t \\ y(m_{t\nu}/m^i) \geq 0}} C_t^i(m_{t\nu}/m^i, s_{t\nu})$$

$$> -\Sigma_i \min_{\substack{m^i \in M_t \\ y(m_{t\nu}/m^i) \geq 0}} C_t^i(m_{t\nu}/m^i, s_{t\nu}) .$$

since $\omega^i \in \text{Int } X_t^i$, $p_{t\nu} \neq 0$, and $\pi_t^j(s_{t\nu}) \geq 0$ as $0 \in Y_t^j$ by (d.1).

Now, if $y(m_{t\nu}) \geq 0$, then $\min_{\substack{m^i \in M_t \\ y(m_{t\nu}/m^i) \geq 0}} C_t^i(m_{t\nu}/m^i, s_{t\nu}) = C_t^i(m_{t\nu}, s_{t\nu})$ and

(*) and (**) imply $\Sigma_i b_t^i > -\Sigma_i C_t^i(m_{t\nu},s_{t\nu}) \geq 0$ contradicting the assumption that $\Sigma_i b_t^i \leq 0$.

But, if $b_t^i \geq 0$ for <u>any</u> i, then $m_{t\nu}^i \geq -(I-1)\mu_{t\nu}^i$ implying that $y(m_{t\nu}) \geq 0$. Thus, $b_t^i(m_{t\nu}^{)i(},s_{t\nu}) < 0$ for <u>every</u> i, which implies that $r_t^i(m_{t\nu},s_{t\nu}) = 0$ and also that

$$(x_{t\nu}^i,m_{t\nu}^i) \in \xi_t^i(m_{t\nu}^{)i(},s_{t\nu};\ w_t^i(s_{t\nu})) \text{ for every } i .$$

However, then

$$C_t^i(m_{t\nu},s_{t\nu}) \geq w_t^i(s_{t\nu}) - \min_{x^i\in X_t^i} p_{t\nu}\cdot x^i > 0 \text{ as above } .$$

Then

$$\Sigma_i C_t^i(m_{t\nu},s_{t\nu}) > 0 \text{ contradicting (*). Hence}$$

$$\Sigma_i b_t^i(m_{t\nu}^{)i(},s_{t\nu}) > 0 \text{ as was to be shown.}$$

(b) Since $\Sigma_i b_t^i > 0$, $b_t^i(m_{t\nu}^{)i(},s_{t\nu}) > 0$ for some i which implies $m_{t\nu}^i \geq -(I-1)\mu_{t\nu}^i$ and thus $y(m_{t\nu}) \geq 0$ as was to be shown.

(c) At the fixed point $\varepsilon_{t\nu}$, for every i, either

(i) $(x_{t\nu},m_{t\nu}) \in \delta_t^i$ which implies by non-satiation (b.1) and convexity (b.3) that $p_{t\nu}\cdot x_{t\nu}^i + C_t^i(m_{t\nu},s_{t\nu}) = w_t^i(s_{t\nu}) - r_t^i(m_{t\nu},s_{t\nu})$, or

(ii) $d_t^i(m_{t\nu}^{)i(},s_{t\nu}) = w_t^i(s_{t\nu}) - r_t^i(m_{t\nu},s_{t\nu})$ which also implies that $p_{t\nu}\cdot x_{t\nu}^i + C_t^i(m_{t\nu},s_{t\nu}) = w_t^i(s_{t\nu}) - r_t^i(m_{t\nu},s_{t\nu})$, or

(iii) $d_t^i(m_{t\nu}^{)i(},s_{t\nu}) > w_t^i(s_{t\nu})$ which implies that

$$p_{t\nu}\cdot x_{t\nu}^i + C_t^i(m_{t\nu},s_{t\nu}) \leq w_t^i(s_{t\nu}) - b_t^i(m_{t\nu}^{)i(},s_{t\nu})$$

where strict inequality holds only if

$$C_t^i(m_{t\nu}/\underline{m}^i,s_{t\nu}) < \min_{\substack{m^i\in M_t \\ y(m_{t\nu}/m^i)\geq 0}} C_t^i(m_{t\nu}/m^i,s_{t\nu})$$

which can occur only if $y_k(m_{t\nu}) < 0$ for some k, a possibility excluded by (b).

Thus,

$$p_{t\nu}\cdot\Sigma_i x_{t\nu}^i + \Sigma_i C_t^i = \Sigma_i w_t^i(s_{t\nu}) - \sum_{b_t^i\geq 0} r_t^i - \sum_{b_t^i<0} b_t^i .$$

Or, using the definitions of w_t^i, r_t^i, and the fact that $\Sigma_i C_t^i = q_{t\nu}\cdot y(m_{t\nu})$,

$$(p_{t\nu}\cdot\Sigma_i(x_{t\nu}^i-\omega^i),\ q_{t\nu}\cdot y(m_{t\nu})) - \bar{s}_{t\nu}\cdot\Sigma_j z_{t\nu}^j = s_{t\nu}\cdot e_{t\nu}$$

$$= \Big|\sum_{b_t^i<0} b_t^i\Big| - \sum_{b_t^i\geq 0} r_t^i$$

$$= 0$$

since $\Sigma_i b_t^i > 0$ by (a).

Thus, since $s_{t\nu} \in \eta_{t\nu}(e_{t\nu})$,

$$0 = s_{t\nu}\cdot e_{t\nu} \geq s\cdot e_{t\nu} \quad \text{for every } s\in S_\nu,$$ completing (c) and the proof of Lemma 3.

(11) For fixed $t > 1/\gamma$, consider the sequence of fixed points $\varepsilon_{t\nu}$ as $\nu \to 0$. Since $\varepsilon_{t\nu}$ for every $\nu < 1$ is in the compact space $\underset{i}{\times}(X^i_t \times M_t) \underset{j}{\times} Z^j_t \times S$ where $S \equiv$ closure $\overset{\circ}{S} = \{s \in \mathbb{R}^{L+K}_+ \mid \|s\| = s \cdot \underline{1} = 1\}$, the sequence has a limit point $\hat{\varepsilon}_t = (<\hat{x}^i_t, \hat{m}^i_t>, <\hat{z}^j_t>, \hat{s}_t)$. It is easy to see that at the limit point $\hat{s}_t \cdot \hat{e}_t = 0$ and $s \cdot \hat{e}_t \leq 0$ for every $s \in S$. Thus, by the same argument as in Debreu [1959], excess demand $\hat{e}_t$ can be shown to be non-positive, i.e. $\hat{e}_t \leq 0$.

Thus, by the assumption of free disposal (d.4), there exists a net aggregate production plan $\tilde{z}_t \in Z$ such that $\tilde{z}_t = \hat{z}_t + \hat{e}_t$. Let $<\tilde{z}^j_t>$ be such that $\tilde{z}^j_t \in Z^j_t$ and $\Sigma_j \tilde{z}^j_t = \tilde{z}_t$.

<u>(12) Lemma 4</u>: Consider the point $\tilde{\varepsilon}_t = (<\hat{x}^i_t, \hat{m}^i_t>, <\tilde{z}^j_t>, \hat{s}_t)$.

(a) $\tilde{e}_t = (\Sigma_i(\hat{x}^i_t - \omega^i), y(\tilde{m}_t)) - \Sigma_j \tilde{z}^j_t = 0$

(b) $\tilde{z}^j_t \in \phi^j(\hat{s}_t)$.

<u>Proof</u>:

(a) is immediate from the definition of $\tilde{z}^j_t$.

(b) since $\hat{s}_t \cdot \hat{e}_t = 0$, by the construction of $\tilde{z}^j_t$, $\hat{s}_t \cdot \hat{z}^j_t = \hat{s} \cdot \tilde{z}^j_t$.

By the u.s.c. of $\phi^j_t(\cdot)$, $\hat{z}^j_t \in \phi^j_t(\hat{s}_t)$. But since $\tilde{\varepsilon}_t$ is attainable, $\tilde{z}^j_t \in Z^j_t$ and as $\hat{s}_t \cdot \hat{z}^j_t = \hat{s}_t \cdot \tilde{z}^j_t$, $\tilde{z}^j_t \in \phi^j_t(\hat{s}_t)$. Also, as $\tilde{z}^j_t$ is in the interior of the cube $B^{L+K}(n(t))$ containing Z^j_t and Z^j is convex, $\tilde{z}^j_t \in \phi^j(\hat{s}_t)$, thus proving (b).

We note also for future use the following easily verified results:

(c) $\pi^j(\hat{s}_t) = \pi^j_t(\hat{s}_t)$

(d) $w^i(\hat{s}_t) = w^i_t(\hat{s}_t)$

(e) $\underset{x^i \in X^i_t}{\mathrm{Min}}\ \hat{p}_t \cdot x^i = \underset{x^i \in X^i}{\mathrm{Min}}\ \hat{p}_t \cdot x^i$.

<u>(D) Discussion</u>: Lemma 4 establishes two of the three properties $\tilde{\varepsilon}_t$ must satisfy to be an equilibrium. Thus, to prove the existence theorem we have remaining only to show that for some $t > 1/\gamma$,

(α) $(\hat{x}^i_t, \hat{m}^i_t) \in \delta^i(\hat{m}_i^{)i(}, \hat{s}_i; w^i(\hat{s}_t))$.

This we will show in three steps. First, we show $(\hat{x}^i_t, \hat{m}^i_t) \in \tilde{\delta}^i_t(\hat{m}_t, \hat{s}_t)$ which requires that $\hat{p}_t \neq 0$. Second, we show that for some t sufficiently large, the compactification bounds on the message space are not binding at the point $\hat{m}_t$, i.e. $-t\underline{1} < \hat{m}^i_t < It\underline{1}$ all i. This will establish that $(\hat{x}^i_t, \hat{m}^i_t) \in \tilde{\delta}^i(\hat{m}_t, \hat{s}_t)$. Finally we show for this sufficiently large t that no consumer is bankrupt or just barely solvent so that $(\hat{x}^i_t, \hat{m}^i_t) \in \delta^i(\hat{m}_t^{)i(}, \hat{s}_t; w^i(\hat{s}_t))$ as required.

<u>(13) Lemma 5</u>: For every $t > 1/\gamma$, at the point $\tilde{\varepsilon}_t$, (a) $\hat{p}_t \neq 0$ and thus, (b) $(\hat{x}^i_t, \hat{m}^i_t) \in \tilde{\delta}^i_t(\hat{m}_t, \hat{s}_t)$.

<u>Proof</u>: (b) follows from (a) by the u.s.c. of $\tilde{\delta}^i_t(\cdot)$ at $\hat{s}_t$ if $\hat{p}_t \neq 0$. To prove (a), suppose $\hat{p}_t = 0$. By Assumption (d.5), for some $z_x \in \mathbb{R}^L$ and $c > 0$, $(z_x, \tilde{z}_{yt} + c\,\underline{1}) \equiv \overset{\circ}{z} \in Z$. Let $<\overset{\circ}{z}{}^j>$ be such that $\overset{\circ}{z}{}^j \in Z^j$ and $\Sigma_j \overset{\circ}{z}{}^j = \overset{\circ}{z}$..

By Lemma 4, $\tilde{z}^j_t \in \phi^j(\hat{s}_t)$ which implies $\hat{s}_t \cdot \tilde{z}^j_t \geq \hat{s}_t \cdot \overset{\circ}{z}^j$. Thus,

$$\hat{s}_t \cdot \Sigma_j \tilde{z}^j_t = \hat{s}_t \cdot \tilde{z}_t = \hat{q} \cdot \tilde{z}_{yt} \geq \hat{s}_t \cdot \Sigma_j \overset{\circ}{z}^j = \hat{s}_t \cdot \overset{\circ}{z} = \hat{q}_t \cdot \overset{\circ}{z}_y$$
$$= \hat{q}_t \cdot (\tilde{z}_{yt} + c\underline{1}) = \hat{q}_t \cdot \tilde{z}_{yt} + c\hat{q}_t \cdot \underline{1} > \hat{q}_t \cdot \tilde{z}_{yt}, \quad \text{contradiction.}$$

(14) Lemma 6: For t sufficiently large,

(a) $-t\underline{1} < \hat{m}^i_t < It\underline{1}$ for every i and

(b) $(\hat{x}^i_t, \hat{m}^i_t) \in \tilde{\delta}^i(\hat{m}_t, \hat{s}_t)$.

Proof: (b) follows from (a) since $\hat{x}^i_t \in \text{int } B^L(n(t))$ (as $\hat{x}^i_t$ is attainable), and $\hat{m}^i_t \in \text{int } M_t$. Thus, the compactification contraints are not binding anywhere. Then by convexity of X^i, preferences, and the budget correspondence $\beta^i(\cdot)$, the result follows.

To show (a), we first show $\hat{m}^i_t < It\,\underline{1}$ for t sufficiently large and then $-t\underline{1} < \hat{m}^i_t$.

Claim 1: $\hat{m}^i_t < It\underline{1}$ for every t sufficiently large, for all i.

Proof: Suppose not. Then for some public good k and consumer i, there is a sequence $t_n \nearrow \infty$ as $n \nearrow \infty$ with $\hat{m}^i_{t_n k} = It_n$ for all n. But, for every t_n

$$y_k(\hat{m}_{t_n}) = \Sigma_h \hat{m}^h_{t_n k} = It_n + \Sigma_{h\neq i} m^h_{t_n k} \geq It_n - (I-1)t_n = t_n \nearrow \infty .$$

But $y_k(\hat{m}_{t_n})$ is bounded above for every t_n since it is attainable. The contradiction establishes Claim 1.

Claim 2: $\hat{m}^i_t > -t$ for every t sufficiently large.

Proof: Suppose not. Then for some public good k and consumer i, there is a sequence $t_n \nearrow \infty$ as $n \nearrow \infty$ with $\hat{m}^i_{t_n k} = -t_n$ for all n. By Lemma 3, $y_k(\hat{m}_{t\nu}) \geq 0$ for every t and ν. Thus $y_k(\hat{m}_{t_n}) \geq 0$ for every t_n. Thus, $\hat{m}^i_{t_n k} < \hat{\mu}^i_{t_n k}$ and hence i is not in a bankrupt or minimum wealth condition. Thus, i is maximizing preferences at each point t_n. Also, $y_k(\hat{m}_{t_n}) \geq 0$ implies $\Sigma_{h\neq i}\hat{m}^h_{t_n k} \geq t_n$ all n and thus, $[(I-1)/I](\hat{m}^i_{t_n k} - \hat{\mu}^i_{t_n k}) \leq -t_n$.

Since $(\hat{x}^i_t, \hat{y}_t) \equiv (\hat{x}^i_t, y(\hat{m}_t)) \in \text{int } B^{L+K}(n(t))$ (by attainability), by convexity of X^i (Assumption a) and non-satiation (Assumption b.1), there is some $\tilde{x}^i_t \in X^i_t$ such that $(\tilde{x}^i_t, \hat{y}_t) \succ_i (\hat{x}^i_t, \hat{y}_t)$.

Furthermore, by the compactness of the attainable set, convexity of X^i, continuity and convexity of preferences (b.2 and b.3), there exists a small strictly positive number $c > 0$, such that for every t

$$(\tilde{x}^i_t, \tilde{y}_t) \equiv (\tilde{x}^i_t, \hat{y}_t/y_{tk} + c) \succ_i (\hat{x}^i_t, \hat{y}_t) \quad \text{and} \quad (\tilde{x}^i_t, \tilde{y}_t) \in X^i_t .$$

Also, since the attainable set is compact, there exists a maximum distance $\bar{\zeta}$ such that if $(\hat{x}^i_t, \hat{y}_t)$ is attainable, there exists some $\tilde{x}^i_t \in X^i$ within $\bar{\zeta}$ of $\hat{x}^i_t$; i.e. $\|\tilde{x}^i_t - \hat{x}^i_t\| \leq \bar{\zeta}$, and $(\tilde{x}^i_t, \tilde{y}_t) \succ_i (\hat{x}^i_t, \hat{y}_t)$.

Now as in (Groves and Ledyard [1977], (6) in Proof of Theorem 4.1), for every t_n, since $(\tilde{x}^i_{t_n}, \tilde{y}_{t_n}) \succ_i (\hat{x}^i_{t_n}, \hat{y}_{t_n})$,

$$\hat{p}_{t_n}\cdot\tilde{x}^i_{t_n} + \hat{C}^i_y\cdot\tilde{y}_{t_n} > \hat{p}_{t_n}\cdot\hat{x}^i_{t_n} + \hat{C}^i_y\cdot\hat{y}_{t_n}$$

where $\hat{C}^i_y \equiv \alpha^i_{t_n}(\hat{s}_{t_n})\hat{q}_{t_n} + \gamma((I-1)/I)(\hat{m}^i_{t_n} - \hat{\mu}^i_{t_n})$. Thus, for every n,

$$0 < \hat{p}_{t_n}\cdot(\tilde{x}^i_{t_n} - \hat{x}^i_{t_n}) + [\alpha^i_{t_n}(\hat{s}_{t_n})\hat{q}_{t_nk} + \gamma(\frac{I-1}{I})(\hat{m}^i_{t_nk} - \hat{\mu}^i_{t_nk})]c$$

and since $((I-1)/I)(\hat{m}^i_{t_nk} - \hat{\mu}^i_{t_nk}) \le -t_n$, and $\alpha^i_{t_n}(\hat{s}_{t_n})\hat{q}_{t_nk} \le 1$

$$\begin{aligned}0 < \hat{p}_{t_n}\cdot(\tilde{x}^i_{t_n} - \hat{x}^i_{t_n}) + [1-\gamma t_n]c &\le \|\tilde{x}^i_{t_n} - \hat{x}^i_{t_n}\| + (1-\gamma t_n)c\\ &\le \bar{\zeta} + (1-\gamma t_n)c\\ &= (\bar{\zeta}+c) - (\gamma c)t_n \quad \text{for all } n\ .\end{aligned}$$

But for t_n sufficiently large $(\bar{\zeta}+c) - (\gamma c)t_n < 0$. Contradiction, thus establishing Claim 2 and Lemma 6. .Q.E.D

(E) Discussion: By Lemmas 4 and 6, for some sufficiently large t, there exists $\tilde{\varepsilon}_t = (\langle\hat{x}^i_t,\hat{m}^i_t\rangle, \langle\tilde{z}^j_t\rangle, \hat{s}_t)$ such that

(α') $\quad (\hat{x}^i_t,\hat{m}^i_t) \in \tilde{\delta}^i(\hat{m}_t,\hat{s}_t)$

(β) $\quad \tilde{z}^j_t \in \phi^j(\hat{s}_t)$

(γ) $\quad (\Sigma_i(\hat{x}^i_t - \omega^i), y(\hat{m}_t)) = \Sigma_j\tilde{z}^j_t$.

Let $\varepsilon^* \equiv (\langle x^{i*},m^{i*}\rangle, \langle z^{j*}\rangle, s^*) = \tilde{\varepsilon}_t$ for the sufficiently large t, and let $y^* = y(m^*)$.

To show now that $(x^{i*},m^{i*}) \in \delta^i(m^{i*},s^*;\ w^i(s^*))$, we need to show that no consumer is bankrupt or in his minimum worth condition at ε^*, i.e. that $b^i(m^*,s^*) > 0$ for all i, which will then mean that no bankruptcy taxes are assessed, i.e. $r^i(m^*,s^*) = 0$ all i. Thus,

$$(x^{i*},m^{i*}) \in \tilde{\delta}^i(m^{i*},s^*;\ w^i(s^*)) \quad \text{if} \quad r^i(m^*,s^*) = 0$$

and

$$(x^{i*},m^{i*}) \in \delta^i(m^{i*},s^*,w^i(s^*)) \text{ if } b^i(m^*,s^*) > 0\ .$$

To show $b^i(m^*,s^*) > 0$ we will show that Assumption (e) will imply that $y^* = y(m^*) \in H$ and then use this fact to show $b^i > 0$.

(15) Lemma 7: At ε^*,

(a) $y^* = y(m^*) \in H$

(b) $q^*\cdot y^* + \frac{1}{\gamma}q^{*2} < \Sigma_i(w^i(s^*) - \underset{x^i\in X^i}{\text{Min}}\ p^*\cdot x^i)$

(c) $b^i(m^{i*},s^*) > 0$ all i .

Proof:

(a) Suppose $y^* \notin H$. Then by Assumption e, there is some attainable allocation $a' \in A$, $y' \in H$ such that, $(x^{i'},y') \succ_i (x^{i*},y^*)$ for every i.

Then, since at least one consumer is strictly solvent by Lemma 3, by the same argument used in Lemma 6 (Claim 2),

$$p^*\cdot(x^{i'} - x^{i*}) + [\alpha^i(s^*)\cdot q^* + \frac{\gamma(I-1)}{I}(m^{i*} - \mu^{i*})]\cdot(y' - y^*) \ge 0$$

every i with strict inequality for at least one i. Thus,

$$p^*\cdot\Sigma_i(x^{i'}-x^{i*})+q^*\cdot(y'-y^*)>0 \quad , \quad \text{or}$$

$$p^*\cdot\Sigma_i x^{i'}+q^*\cdot y' > p^*\cdot\Sigma_i x^{i*}+q^*\cdot y^* = \Sigma_i w^i(s^*) \text{ (by Lemma 3)}$$

$$\Sigma_i w^i(s^*) = \Sigma_i p^*\cdot\omega^i + \Sigma_j\Sigma_i\theta^{ij}\pi^j(s^*) \geq \Sigma_i p^*\cdot\omega^i + \Sigma_j s^* z^{j'}$$

$$= \Sigma_i p^*\cdot\omega^i + s^*[\Sigma_i(x^{i'}-\omega^i),y'] = \Sigma_i p^*\cdot x^{i'} + q^*\cdot y' \; .$$

Contradiction; thus (a) is established.

(b) $y^*\in H$ implies there exists $a'\in A$, $y' = y^* + (1/\gamma)\underline{1}$ and $(\Sigma_i(x^{i'}-\omega^i),y') = \Sigma_j z^{j'}$.

Thus

$$p^*\cdot\Sigma_i x^{i'}+q^*\cdot y' = p^*\cdot\Sigma_i\omega^i+\Sigma_j s^*\cdot z^{j'} \leq p^*\cdot\Sigma_i\omega^i+\Sigma_i\Sigma_j\theta^{ij}\pi^j(s^*)$$

$$= \Sigma_i w_i(s^*) \; .$$

Hence

$$q^*\cdot y' = q^*\cdot y^* + \frac{1}{\gamma}q^*\cdot\underline{1} \leq \Sigma_i w^i(s^*) - p^*\cdot\Sigma_i x^{i'} \leq \Sigma_i\Big(w^i(s^*) - \min_{x^i\in X^i} p^*\cdot x^i\Big).$$

Since $q^*_k < 1$ all k,

$$q^*\cdot y^* + \frac{1}{\gamma}q^{*2} < q^*\cdot y^*+\frac{1}{\gamma} q^*\cdot\underline{1} \le \Sigma_i\Big(w^i(s^*) - \min_{x^i\in X^i} p^*\cdot x^i\Big)$$

$$\text{if } q^*\neq 0$$

and

$$0 = q^*\cdot y^* + \frac{1}{\gamma} q^{*2} < \Sigma_i\Big(w^i(s^*) - \min_{x^i\in X^i}p^*\cdot x^i\Big) \quad \text{if } q^*=0 \; ,$$

since $\omega^i\in \text{Int } X^i$ and $p^*\neq 0$. Thus (b) is proved.

(c) By definition

$$b^i(m^{)i(*},s^*) = \Big(w_i(s^*) - \min_{x^i\in X^i} p^*\cdot x^i\Big) - \min_{\substack{m^i\in M\\ y(m^*/m^i)\geq 0}} C^i(m^*/m^i,s^*) \; .$$

By (b) and the definition of $\alpha^i(s^*)$,

(A) $$b^i(m^{)i(*},s^*) > \alpha^i(s^*)\Big[q^*\cdot y^*+\frac{1}{\gamma} q^{*2}\Big] - \min_{\substack{m^i\in M\\ y(m^*/m^i)\geq 0}} C^i(m^*/m^i,s^*) \; .$$

Now, since $y^* = y(m^*)\geq 0$ and $(x^{i*},m^{i*})\in\tilde\delta^i(m^*,s^*)$,

(B) if $b^i(m^{)i(*},s^*)\le 0$, then $C^i(m^*,s^*) = \displaystyle\min_{\substack{m^i\\ y(m^*/m^i)\geq 0}} C^i(m^*/m^i,s^*)$ which implies that $m^{i*}\le\mu^{i*}$ and $\alpha^i(s^*)q^* + \gamma((I-1)/I)(m^{i*}-\mu^{i*}) \geq 0$. Thus

$$0 \geq (m^{i*}-\mu^{i*}) \geq -\frac{\alpha^i(s^*)I}{\gamma(I-1)}q^* \text{ which implies } (m^{i*}-\mu^{i*})^2 \leq \frac{(\alpha^i(s^*))^2 I^2}{\gamma^2(I-1)}q^{*2} \; .$$

Hence

$$C^i(m^*,s^*) = \alpha^i(s^*)q^* \cdot y^* + \frac{\gamma}{2}\left[\frac{I}{I-1}(m^{i*} - \mu^{i*})^2 - (\sigma^{i*})^2\right]$$

$$\leq \alpha^i(s^*)q^* \cdot y^* + \frac{I}{2\gamma(I-1)}(\alpha^i(s^*))^2 q^{*2}$$

$$\text{(C)} \qquad \leq \alpha^i(s^*)\left[q^* \cdot y^* + \frac{\alpha^i(s^*)I}{2\gamma(I-1)}q^{*2}\right] \leq \alpha^i(s^*)\left[q^* \cdot y^* + \frac{1}{\gamma}q^{*2}\right] \quad .$$

Combining (A), (B), and (C), if $b^i(m^{)i(*},s^*) \leq 0$, then

$$0 \geq b^i(m^{)i(*},s^*) > \alpha^i(s^*)(q^* \cdot y^* + \frac{1}{\gamma}q^{*2}) - C^i(m^*,s^*) \geq 0 \quad .$$

Contradiction. Thus, $b^i(m^{)i(*},s^*) > 0$, all i. Q.E.D.

Footnotes

1/ See Groves and Ledyard [1977], Remark 4.3, p. 800.

2/ Both the mechanisms of Hurwicz [1976] and Walker [1977] have the property that Lindahl allocations are (Nash) equilibrium allocations. Thus, the conjecture is true for their mechanisms. However, their mechanisms have some other, possibly less desireable, properties not possessed by our mechanism.

3/ Lindahl allocations never leave a consumer worse off than at his initial endowment.

4/ The notation $\langle x^i \rangle$ denotes the I-tuple $(x^1,\ldots,x^I)$; similarly for $\langle z^j \rangle$, $\langle \theta^{ij} \rangle$, etc.

5/ As the allocation rule $y(\cdot)$ for our government depends only on joint messages m (see (2.2) below), henceforth $y(m,s) = y(m)$. Also, throughout we use the notation

$$m^{)i(} \equiv (m^1,\ldots,m^{i-1},m^{i+1},\ldots,m^I)$$

$$m/\bar{m}^i \equiv (m^1,\ldots,m^{i-1},\bar{m}^i, m^{i+1},\ldots,m^I) \quad .$$

6/ In our earlier paper we called this government the Optimal government referring to the property that competitive equilibria under this government are Pareto-optimal. However, as other mechanisms (see introduction) also have this property, the label "Optimal" seems no longer appropriate and possibly misleading.

7/ This section may be skipped without loss of continuity in the formal analysis of the paper.

8/ This requires verifying that $\Sigma_i C^i(m) = y(m)$ for every $m \in M^I$.

9/ Michael Rothschild brought this point to our attention in some class notes [1976] which also contain an analysis of our mechanism in the simple two good neo-classical economy.

10/ The scalar $1/\gamma$ can be replaced by the smaller scalar $I/2\gamma(I - 1)$.

11/ Assumption A.4 and the restriction of Theorem 3 on the parameters α^i impose joint restrictions on the economies and parameters of the mechanism. This type of joint restriction seems unavoidable when potential bankruptcy is present. An example of a similar assumption can be found in Debreu [1962] for private goods only competitive models when initial endowments are not required to be in consumers' consumption sets.

12/ Our standard assumptions are nearly identical to Debreu's [1959] for the private goods only economy and are quite similar to Milleron's [1972] or Foley's [1970] assumptions for proving the existence of a Lindahl equilibrium.

13/ Both Milleron [1972] and Foley [1970] in proving the existence of Lindahl equilibria need assumptions to rule out private goods prices being driven to zero. Foley assumes the aggregate technology set of the economy is a cone and that every public good is producible. Milleron assumes initial endowments (of private <u>and</u> public goods) are in the interior of consumers' consumption sets and that there is an attainable allocation such that each producer is in the interior of his production sets. These assumptions imply our Assumption (d.5).

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